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NUMERICAL SOLUTION
OF
ELLIPTIC BOUNDARY VALUE PROBLEMS
BY
SPLINE FUNCTIONS

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A numerical method for solving linear, two-dimensional elliptic boundary value problems is presented. The method is essentially the Ritz procedure which uses polynomial spline functions to approximate the exact solution. The spline functions are constructed by defining a polynomial function over each of a set of disjoint subdomains, and imposing certain compatibility conditions along common boundaries between subdomains. The main advantage of the method is that it does not even require the continuity of the spline functions across the boundaries between subdomains. Therefore it is easy to construct classes of spline functions which will produce any specified rate of convergence.

1. INTRODUCTION

During the last few years polynomial splines have been used to obtain numerical solutions of specific elliptic boundary value problems [2,3,4]. This paper presents a general formulation of this type for solving two-dimensional elliptic boundary value problems. The method is essentially the Ritz procedure applied to a finite dimensional space of polynomial spline functions. The formulation can be extended to higher dimensional cases and to simultaneous systems.

We denote by G a bounded open domain in the Euclidean plane with boundary ∂G and closure \bar{G} . For a function $u \in C^j(\bar{G})$ we denote by $|||u|||_{j,G}$ the maximum norm $\max_{|\alpha| \leq j} \max_{\bar{x} \in \bar{G}} |D^\alpha u(\bar{x})|$ and denote by $||u||_{j,G}$ the norm $\left(\int_G \sum_{|\alpha| \leq j} |D^\alpha u(\bar{x})|^2 d\bar{x} \right)^{1/2}$. Here $\alpha = (\alpha_1, \alpha_2)$ is the multiple index of length $|\alpha| = \alpha_1 + \alpha_2$ and D^α is the corresponding partial derivative. We denote by $|u|_{j,G}$ the semi-norm $\left(\int_G \sum_{|\alpha|=j} |D^\alpha u|^2 d\bar{x} \right)^{1/2}$. Let $L = \sum_{|\alpha| \leq 2m} b_\alpha(\bar{x}) D^\alpha$ be a real strongly elliptic operator of order $2m$ defined in \bar{G} . Suppose that $b_\alpha \in C^\infty(\bar{G})$ and G is of class C^∞ . Let the bilinear form associated with the operator L be given by

$$(1.1) \quad B(u, v) = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} \int_G \left\{ (D^\alpha u) (b_{\alpha\beta} D^\beta v) \right\} d\bar{x}$$

where $u, v \in C^\infty(\bar{G})$. Then $L = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} b_{\alpha\beta} D^\alpha D^\beta$. We assume that $B(\cdot, \cdot)$

satisfies the Garding's inequality in the following form:

$$(1.2) \quad |B(\varphi, \varphi)| \geq C |\varphi|_{m, G}^2 \quad \text{for all } \varphi \in C^\infty(\bar{G}')$$

where G' is any open subset of G .

For a given $f \in C^\infty(\bar{G})$, let $\bar{u} \in C_0^\infty(\bar{G})$ be the infinitely differentiable solution satisfying the equation $L\bar{u} = f$ and satisfying zero Dirichlet data on the boundary. Then

$$(1.3) \quad B(\varphi, \bar{u}) = \int_G \varphi f dx \quad \forall \varphi \in C_0^\infty(G)$$

2. TWO-DIMENSIONAL SPLINE FUNCTIONS

Subdivide the domain \bar{G} by a grid h into a finite number of subdomains such that each subdomain is a polygon with infinitely differentiable curvilinear sides. We call any such subdomain a cell. Let $\{S_i\}$ be the set of grid lines forming the sides of cells. We require that the interior of any grid line S_i

must not contain a vertex. Let $\sup_i (\text{Length of } S_i) = h$. We denote by $\{G_i\}$ the disjoint collection of interiors of the cells and let $G_0 = \bigcup_i G_i$.

Assume that each grid line is given by an equation of the form

$$(1.4) \quad \begin{aligned} x &= f_i(\xi_i) \\ y &= g_i(\xi_i) \end{aligned}$$

where ξ_i is the distance along the grid line S_i measured from one of the vertices which it connects. The unit tangent vector $\vec{t}_i = (t_1, t_2)$ at a point on the grid line and the unit normal $\vec{n}_i = (-t_2, t_1)$ are defined by the equation:

$$(1.5) \quad t_1 = \frac{df_i/d\xi_i}{\left[\left(\frac{df_i}{d\xi_i} \right)^2 + \left(\frac{dg_i}{d\xi_i} \right)^2 \right]^{1/2}}$$

$$t_2 = \frac{dg_i/d\xi_i}{\left[\left(\frac{df_i}{d\xi_i} \right)^2 + \left(\frac{dg_i}{d\xi_i} \right)^2 \right]^{1/2}}$$

If $u^{(i)} \in C^\infty(\bar{G}_1)$, let $D_{\alpha, r}^{(j)} u^{(i)}$ denote the r -th derivative of $u^{(i)}$ in the

direction of the normal \vec{n}_i , defined if S_j is one of the sides of G_j . For every cell, we define the following quantities:

$$(1.6) \quad u_{kr}^{ij} = \frac{1}{h^{k+1}} \int_{S_j} \epsilon_j^k D_{n,r}^{(j)} u^{(i)} d\epsilon_j \quad \begin{array}{l} \text{defined for all } k \text{ and } r \\ \text{whenever} \\ 0 \leq r \leq m-1 \\ \text{and } 0 \leq k \leq p+r-(m-1) \end{array}$$

where p is some fixed integer ≥ 0 and independent of h . Also if w is a vertex of a cell G_i , define $u_\alpha^{i,w} = D_\alpha u^{(i)}(w)$ for all α such that $0 \leq |\alpha| \leq m-2$.

We define a class $\mathcal{K}(h, p)$ of functions with domain G as follows: Let (x_i, y_i) denote the local coordinates of a cell G_i , obtained by translation of the origin to a point in G_i . If $u \in \mathcal{K}(h, p)$, then $u^{(i)} = u$ restricted to G_i is given by an equation of the form

$$(1.7) \quad u^{(i)} = a_{00} + a_{10} x_i + a_{01} y_i + \dots + a_{0, n_i} y_i^{n_i}$$

such that the following compatibility conditions are satisfied:

(i) For all k and r and whenever defined

$$(1.8) \quad u_{kr}^{ij} = u_{kr}^{i'j} \quad \text{if } S_j = \partial G_i \cap \partial G_{i'},$$

$$\text{and } u_{kr}^{ij} = 0 \quad \text{if } S_j = \partial G \cap \partial G_i$$

(ii) At a vertex w , for all $|\alpha| \leq m-2$

$$(1.9) \quad u_{\alpha}^{i,w} = u_{\alpha}^{i',w} \quad \text{if } w \text{ belongs to both } G_i \text{ and } G_{i'},$$

$$\text{and } u_{\alpha}^{i,w} = 0 \quad \text{if the } w \notin \partial G$$

We also require that

$$(1.10) \quad n_i \geq m + p$$

Now we impose the following conditions on the class of admissible grids:

First we introduce $m+n$ -dimensional coordinates $X_i = x_i/h$, $Y_i = y_i/h$, $\bar{X}_i = \bar{x}_i/h$

and quantities $A_{jk} = a_{jk} h^{j+k}$, $U_{kr}^{ij} = u_{kr}^{ij} h^r$ and $U_{\alpha}^{i,w} = u_{\alpha}^{i,w} h^{|\alpha|}$ so that

$$(1.11) \quad u^{(i)} = A_{00} + A_{10} X_i + \dots + A_{0n_i} Y_i^{n_i}$$

$$(1.12) \quad U_{kr}^{ij} = \int_{S_j} \sum_j^k (h^r D_{n,r}^{(j)}) u^{(i)} dS_j, \quad U_{\alpha}^{i,w} = (h|\alpha|D_{\alpha}) u^i(w)$$

For a given cell G_i , let V_i be the vector space spanned by the quantities

U_{kr}^{ij} and $U_{\alpha}^{i,w}$ and let A_i be the vector space spanned by the coefficients A_{jk} .

Then we have a linear transformation

$$(1.13) \quad T_i : A_i \rightarrow V_i$$

We choose n_i sufficiently large so that T_i is onto. Now let K_i be the kernel of T_i and let \bar{A}_i be the complement of K_i . Then T_i induces an isomorphism

$$(1.14) \quad \bar{T}_i : \bar{A}_i \rightarrow V_i$$

We require that : (a) Values of n_i are bounded uniformly with respect to all admissible grids and the absolute values of the eigenvalues of \bar{T}_i are bounded away from zero uniformly with respect to all admissible grids. (b) There exists a positive constant δ_i such that for any admissible grid,

$$\inf_i \frac{\text{length of } S_i}{h} \geq \delta_i$$

(c) There exists a positive constant δ_2 such that for every admissible grid, and for every cell, there exists a point (x_0, y_0) in that cell such that the

square $\{(x, y) \mid |x - x_0| \leq \delta_2 h, |y - y_0| \leq \delta_2 h\}$ is contained in the cell. (d) Modulo

translation, rotation and scale factor, grid lines S_1 are restricted to a

finite number of shapes. (e) There exists a constant C_0 such that the number of cells that any straight line through G intersects is less than C_0/h .

If the cells are triangular or rectangular in shape with straight sides and if they satisfy conditions (b) and (c), then it is possible to choose a single integer n for n_i in Eq. (1.7) such that condition (a) is satisfied for all such cells. To show this for triangular cells, consider a cell G_1 in the form of an isosceles right-angled triangle with hypotenuse h units long and with origin of the local coordinate system at one of the vertices. Since the transformation T_1 for this particular cell does not depend upon h , we can find an integer n for n_1 in Eq. (1.7) such that condition (a) is satisfied for all values of h . Now consider any other triangular cell G_i which satisfies conditions (b) and (c). Set $n_i = n$. Without loss of generality we can assume that G_i has the origin of the local coordinate system at one of its vertices. There

exists a non-singular linear transformation S mapping G_i onto G_i . Now specification of the quantities $u_{\alpha}^{i,w}$ and u_{kr}^{ij} (if defined) implies specification of all quantities of the form

$$\frac{1}{h^{k+1}} \int_{S_j} \xi_j^k D_{u,r}^{(j)} D_{t,r'}^{(j)} u^{(i)} d\xi_i \quad \begin{array}{l} 0 \leq r+r' \leq m-1 \\ r' > 0 \\ 0 \leq k \leq p+(r+r')-(m-1) \end{array}$$

where $D_{t,r'}^{(j)} u^{(i)}$ denotes the r' -th derivative of $u^{(i)}$ in the direction of the tangent, defined if S_j is one of the sides of G_i .

Therefore S induces an isomorphism

$$(1.17) \quad S_2 : V_i \simeq V_i.$$

Also S induces an isomorphism

$$(1.18) \quad S_1 : A_i \simeq A_i.$$

Therefore we have a commutative diagram

$$(1.19) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K_i & \longrightarrow & A & \xrightarrow{T_1} & V_i \longrightarrow 0 \\ & & \downarrow & & \downarrow S_1 & & \downarrow S_2 \\ 0 & \longrightarrow & K_i & \longrightarrow & A_i & \xrightarrow{T_i} & V_i \longrightarrow 0 \end{array}$$

This, in turn, implies a commutative diagram

$$(1.20) \quad \begin{array}{ccc} \bar{A}_i & \xrightarrow{\bar{T}_i} & V_i \\ \downarrow S_1 & & \downarrow S_2 \\ \bar{A}_{i'} & \xrightarrow{\bar{T}_{i'}} & V_{i'} \end{array}$$

Therefore $\bar{T}_{i'}$ is nonsingular. Now we claim that condition (a) is satisfied.

For, if not, there exists a sequence of triangular cells $\{G_k\}$ such that the corresponding sequence of the lowest absolute eigenvalues of the transformations \bar{T}_k tends to zero. But any triangular cell is specified by six quantities, namely the non-dimensional coordinates of its vertices. Therefore the sequence of triangles can be embedded as a closed subspace lying in the closed annulus in \mathbb{R}^6 , lying between the unit sphere and the sphere of radius δ_1 . Since the annulus is compact, the sequence G_k has an accumulation point. By continuity, the triangular cell G_{k_0} corresponding to the accumulation point satisfies conditions (b) and (c) and has singular \bar{T}_{k_0} which contradicts (1.20).

In the same fashion, we can choose a single integer in Eq. (1.7) which will satisfy condition (a) for all rectangular cells provided that conditions (b) and (c) are satisfied.

For practical applications, it suffices to segment a domain using triangular and rectangular cells as well as a finite number of special shapes to handle a curved boundary.

The vector space $\mathcal{F}(h,p)$ is constructed as follows. For each cell G_i , we have an isomorphism $A_i \cong V_i \oplus K_i$. Then a function $u \in \mathcal{F}(h,p)$ is defined by specifying the quantities

$$u_{kr}^j = u_{kr}^{ij} = u_{kr}^{i'j} \quad \text{for each grid line not in } \partial G$$

$$u_{\alpha}^w = u_{\alpha}^{i,w} = u_{\alpha}^{i',w} = \dots \quad \text{for each vertex not in } \partial G$$

and an element of $K^{(i)}$ for each cell G_i .

We require that

$$u_{kr}^j = 0 \text{ if } S_j \text{ is in } \partial G$$

$$\text{and } u_{\alpha}^{i,w} = 0 \text{ if } w \text{ is in } \partial G$$

We extend the (semi) norms $\|\cdot\|, \|\cdot\|_1, \|\cdot\|_2$ and the bilinear form $B(\cdot, \cdot)$ to $\mathcal{F}(\cdot, p)$ by restricting the integrals to G_0 .

Let V be a vector space spanned by the quantities u_{kr}^j, u_α^w (S_j and w not in ∂G) with the basis $\{e_i\}$. Let $K = \bigoplus_i K_i$ with basis $\{k_i\}$. Let $W = V \oplus K$. Then we have an isomorphism $I : \mathcal{F}(\cdot, p) \cong W$. The images of e_i and k_j under I form a basis for $\mathcal{F}(\cdot, p)$. Denote this basis by $\{u_i\}$.

3. FORMULATION

We seek a solution $\hat{u}_\varepsilon \in \mathcal{F}(\cdot, p)$ such that the equation

$$(1.21) \quad B(\varphi, \hat{u}) = \int_{G_0} \varphi f d\bar{x}$$

is satisfied for all $\varphi \in \mathcal{F}(\cdot, p)$. We show that Eq. (1.21) has a unique solution

and that $D^\alpha (\hat{u} - \bar{u}) = O(h^{p+\frac{1}{2}})$ for $0 \leq |\alpha| \leq m-1$.

For a numerical solution of Eq. (1.21) we reduce it to an equivalent matrix

equation as follows. We want to find $\hat{v}_\varepsilon \in W$ such that $\hat{u} = I^{-1}(\hat{v})$ satisfies

Eq. (1.21). Carrying out the integrations, $B(\varphi, \hat{u}) = (v, \bar{B} \hat{v})$ where $v = I(\varphi)$,

(\cdot, \cdot) is the inner product and \bar{B} is a matrix (\bar{B}_{ij}) where $\bar{B}_{ij} = B(u_i, u_j)$. Also

$\int_{\Omega} F d\bar{x} = (v, F)$ where $F \in V$ and its i -th component is given by $\int u_i F d\bar{x}$. Therefore

Eq. (1.21) is equivalent to the matrix equation:

$$(1.22) \quad \bar{B} \bar{v} = F$$

If within each cell, we approximate the coefficients $b_{\alpha\beta}$ and the function f by polynomials use of quadrature formulas for integration of polynomials over simplexes [5] simplifies the computation of \bar{B} . The computation is carried out by computing $B(u_i, u_j)$ cell by cell.

4. CONVERGENCE

In the following C_1, C_2, \dots denote positive numbers, the choice of which does not depend upon h .

Lemma 1: Let $u \in \mathcal{F}(h, p)$. Let S_j be a grid line contained in $\partial G_1 \cap \partial G_1$, and let $u^i = u$ restricted to G_1 and $u^{i'} = u$ restricted to G_1 . Then for $|\alpha| \leq m-1$

$$(1.23) \quad \sup_{(x,y) \in S_j} |D_{\alpha} u^i(x,y) - D_{\alpha} u^{i'}(x,y)| \leq C_1 h^{m-1-|\alpha|} (|u|_{m,G_1} + |u|_{m,G_1})$$

If $S_j = \partial G_i \cap \partial G$, then for $|\alpha| \leq m-1$

$$(1.24) \quad \sup_{(x,y) \in S_j} |D_{\alpha}^{u^1}(x,y)| \leq C_1 h^{m-1-|\alpha|} |u|_{m,G_i}$$

Proof: Suppose that S_j connects vertices w_1 and w_2 . Without loss of generality

we can assume that w_1 is the origin and that the x-axis passes through w_2 . Let

λ_j be the length of S_j . Now consider a subspace \mathcal{Q}_0 of the vector space A_1

spanned by $\{A_{00}, A_{10}, A_{01}, \dots, A_{0,m-1}\}$ and a subspace \mathcal{B}_0 of V_1 spanned by

$\{U_{\alpha}^{1,w_1}\}_{|\alpha| \leq m-2}, \{U_{\alpha}^{1,w_2}\}_{|\alpha| \leq m-2}$ and $U_{0,m-1}^{1j}$. (If $m = 1$, then \mathcal{B}_0 is one-

dimensional, spanned by $U_{0,0}^{1j}$). Let $j : \mathcal{Q}_0 \rightarrow A_1$ be the inclusion map and

$p : V_1 \rightarrow \mathcal{B}_0$ be the projection map. Then it is easy to check that the map

$\tau_1 : \mathcal{Q}_0 \rightarrow \mathcal{B}_0$ given by the composition $\text{pot}_1 \circ j$ is an isomorphism and therefore

$\mathcal{Q}_0 \subset \bar{A}_1$. Moreover since grid lines are restricted to a finite number of

shapes and $\lambda_j/h \in [\delta_1, 1]$, it follows that the absolute values of the eigen-

values of τ_1 are bounded away from zero uniformly with respect to h .

Let \mathcal{B}_1 be the complement of \mathcal{B}_0 in V_1 and \mathcal{Q}_1 be the complement of \mathcal{Q}_0

in \bar{A}_1 . Let $\mathcal{Q}'_0, \mathcal{Q}'_1, \mathcal{B}'_0, \mathcal{B}'_1$ be the corresponding spaces defined for the cell G_i .

Let $u \in \mathcal{H}(h, p)$ and let $w \in W$ and $k \in K$ be the vectors corresponding to u .

Let the projection of w onto V_i be v and let $v = v_0 \oplus v_1 \in \mathcal{V}_0 \oplus \mathcal{V}_1$ where $v_0 \in \mathcal{V}_0$

and $v_1 \in \mathcal{V}_1$. Let $a_0 = \tau_i^{-1}(v_0)$ and let the unique polynomial corresponding to a_0

be $P = \beta_{00} + \beta_{10} x_i + \dots + \beta_{0,m-1} Y_i^{m-1}$. Similarly, the corresponding vector

$v'_0 \in \mathcal{V}_0$ determines a unique polynomial $P' = \beta'_{00} + \beta'_{10} x_i + \dots + \beta'_{0,m-1} Y_i^{m-1}$. But

$v_0 = v'_0$. Therefore $P = P'$.

Let $d = \bar{\tau}_i^{-1}((a_0 \oplus v_1))$ and $d = (d_0 \oplus d_1) \in \mathcal{Q}_0 \oplus \mathcal{Q}_1$ where $d_0 \in \mathcal{Q}_0$ and $d_1 \in \mathcal{Q}_1$.

Let $\tau_q^* = \bar{\tau}_i$ restricted to \mathcal{Q}_q , $q = 0, 1$. Then $\text{po}\tau_0^* + \text{po}\tau_1^*$ is the zero map and

$\text{po}\tau_0^* = \tau_i$. Therefore $d_0 = -\tau_i^{-1}(\tau_1^*(d_1))$ which implies that if $d_0 = (Y_{00},$

$Y_{10}, \dots, Y_{0,m-1})$ and $d_1 = (Y_{m,0}, \dots, Y_{0,n_1})$. Then $|Y_\alpha| \leq C_2 \max_{|\beta| \leq m} |Y_\beta|$ for

$|\alpha| \leq m$.

We define the following norms:

For $q = (q_{00}, q_{10}, \dots, q_{0,m-1}) \in \mathcal{Q}_0$ define the norm $\|q\| = \max_{i,j} |q_{ij}|$.

For $r = (r_{00}, r_{10}, \dots, r_{0,n_1}) \in \mathcal{Q}_1$ define the norm $\|r\| = \max_{(i,j)} |r_{ij}|$.

For $s = (s_{m,0}, \dots, s_{0,n_1}) \in K_i$, define the norm $\|s\| = \max_{(i,j)} |s_{ij}|$.

For $rs = z \in C_1 \oplus K_1$ define the norm $||z|| = ||r|| + ||s||$.

Then $||d_o|| \leq C_2 ||d_1||$. Similarly $||d'_o|| \leq C_2 ||d'_1||$ for the cell G_1 . Now

$\bar{T}_1[(a_o + d_o) \oplus d_1] = v$. Let the projection of k onto K_1 be the vector

$k_1 = (k_{m,o}, \dots, k_{o,n_1})$ and let $d_1 \oplus k_1 = \ell_1$ and $d'_1 \oplus k'_1 = \ell'_1$. Then

$$u^{(i)} = (\beta_{oo} + \gamma_{oo}) + \dots + (\beta_{o,m-1} + \gamma_{o,m-1}) y_i^{m-1} + (k_{m,o} + \gamma_{m,o}) x_i^m + \dots + (k_{o,n_1} + \gamma_{o,n_1}) y_i^{n_1}$$

$$u^{(i')} = (\beta'_{oo} + \gamma'_{oo}) + \dots + (\beta'_{o,m-1} + \gamma'_{o,m-1}) y_i^{m-1} + (k'_{m,o} + \gamma'_{m,o}) x_i^m + \dots + (k'_{o,n_1} + \gamma'_{o,n_1}) y_i^{n_1}$$

$$u^{(i)} - u^{(i')} = (\gamma_{oo} - \gamma'_{oo}) + \dots + (\gamma_{o,m-1} - \gamma'_{o,m-1}) y_i^{m-1} + (k_{m,o} + \gamma_{m,o} - k'_{m,o} - \gamma'_{m,o}) x_i^m + \dots$$

Therefore $\sup_{(x,y) \in S_j} |D_\alpha u^{(i)}(x,y) - D_\alpha u^{(i')}(x,y)| \leq \frac{C_3}{h^{|\alpha|}} (||\ell_1|| + ||\ell'_1||)$

Now we claim that

$$(1.25) \quad ||\ell_1|| \leq C_4 h^{m-1} |u|_{m,G_1}$$

To prove this observe that

$$h^{m-1} |u|_{m,G_1} = \left[\int \sum_{\alpha_1 + \alpha_2 = m} \left(\frac{\partial^{m,u(i)}}{\partial x_1^{\alpha_1} \partial y_1^{\alpha_2}} \right)^2 dx_i dy_i \right]^{1/2}$$

The quadratic form $[\cdot , \cdot] = \left[\sum_{\alpha_1 + \alpha_2 = m} \left(\frac{\partial^m u(i)}{\partial x_1^{\alpha_1} \partial y_1^{\alpha_2}} \right)^2 dx_1 dy_1 \right]$

is positive definite on the linear space $\mathcal{C}_1 \otimes K_1$. Therefore

$$\inf_{0 \neq z \in \mathcal{C}_1 \otimes K_1} \frac{1/2 [z, z]}{\|z\|^2} = \lambda_1 \neq 0. \quad \lambda_1 \text{ is a function of } i \text{ and } h. \quad \text{Suppose } \inf_{i, h} \lambda_1 = 0.$$

Then, there exists a sequence of cells $\{G_k\}$ such that the corresponding

sequence $\lambda_k \rightarrow 0$. Now each G_k contains a point (x_k, y_k) and the square

$\sigma_k = \{(x, y) \mid |x - x_k| \leq h\delta_2, |y - y_k| \leq h\delta_2\}$. Let $[\cdot, \cdot]'$ denote the bilinear form $[\cdot , \cdot]$

restricted to the square σ_k . Then there exists a sequence of points (X_k, Y_k) ,

$|X_k| \leq 1 - \delta_2, |Y_k| \leq 1 - \delta_2$ and a corresponding sequence $\{z_k\}$ such that $\|z_k\| = 1$ and

$[z_k, z_k]' \rightarrow 0$. By compactness, there exists a subsequence $(X_{k_j}, Y_{k_j}, z_{k_j})$ con-

verging to $(X_{k_0}, Y_{k_0}, z_{k_0})$ in $\mathbb{R}^2 \otimes \mathcal{C}_1 \otimes K_1$ such that $\|z_{k_0}\| = 1$ and $[z_{k_0}, z_{k_0}]' = 0$

which is a contradiction of the positive-definiteness of $[\cdot , \cdot]'$. This proves

the claim. Inequality (1.23) follows. The proof of Inequality (1.24) is similar.

Lemma 2: Let $u \in \mathcal{H}(h, p)$. Then

$$(1.26) \quad \| \|u\| \|_{m-1, G_0} \leq \frac{C_3}{h^{\frac{1}{2}}} \|u\|_{m, G_0}$$

Proof: Without loss of generality we may assume that none of the grid lines

is parallel to the coordinate axes. Define a function u' on \mathbb{R}^2 by letting

$u' = u$ on G_0 and $u' = 0$ elsewhere. For any line $y = y_0$ intersecting G_0 , pick

points (x_a, y_0) and (x_b, y_0) in ∂G such that all points having coordinates

(x, y_0) and lying in G be between (x_a, y_0) and (x_b, y_0) . For a fixed $\alpha = (\alpha_1, \alpha_2)$,

$|\alpha| = m-1$, define a function

$$J_\alpha(t) = \begin{cases} D_\alpha u'(x_a + t(x_b - x_a), y_0) & \text{if } (x_a + t(x_b - x_a), y_0) \in G_0 \\ 0 & \text{otherwise} \end{cases}$$

Then $J_\alpha(t)$ is piecewise continuous and differentiable over open segments

$(-\infty, t_0)$, (t_0, t_1) , ..., (t_k, ∞) , where $t_0 = 0$, $t_k = 1$.

$$\text{Define } J_\alpha(t_1+) = \lim_{\substack{t \rightarrow t_1 \\ t-t_1 > 0}} J_\alpha(t)$$

$$\text{and } J_\alpha(t_1-) = \lim_{\substack{t \rightarrow t_1 \\ t-t_1 < 0}} J_\alpha(t)$$

Let $J'_\alpha(t) = \frac{dJ_\alpha}{dt}$ for t in one of the open segments listed above. Then

$$J_{\alpha}(t_{i+1}^-) = J_{\alpha}(t_i^+) + \int_{t_i}^{t_{i+1}} J'_{\alpha}(\tau) d\tau \text{ for } i=0, 1, \dots, k-1. \quad \therefore \text{For any point } t \in (t_i, t_{i+1}),$$

$$J_{\alpha}(t) = J_{\alpha}(t_i^+) + \int_{t_i}^t J'_{\alpha}(\tau) d\tau$$

$$|J_{\alpha}(t)|^2 = \left[\sum_{j=0}^i \{J_{\alpha}(t_j^+) - J_{\alpha}(t_j^-)\} + \int_{t_0}^t J'_{\alpha}(\tau) d\tau \right]^2$$

$$\leq 2 \left[\sum_{j=0}^i \{J_{\alpha}(t_j^+) - J_{\alpha}(t_j^-)\} \right]^2 + 2 \left[\int_{t_0}^t J'_{\alpha}(\tau) d\tau \right]^2$$

$$\leq 2i \left[\sum_{j=0}^i \{J_{\alpha}(t_j^+) - J_{\alpha}(t_j^-)\}^2 \right] + 2t \left[\int_{t_0}^t |J'_{\alpha}(\tau)|^2 d\tau \right]$$

by Cauchy-Schwarz Inequality. From Inequalities (1.23) and (1.24),

$$\sum \{J_{\alpha}(t_j^+) - J_{\alpha}(t_j^-)\}^2 \leq 2C_5 \sum_k |u|_{m, G_k}^2 \leq 2C_5 |u|_{m, G_0}^2$$

$$\text{From Inequality (1.25),} \quad |J'_{\alpha}(\tau)|^2 \leq \frac{C_6}{h^2} |u|_{m, G_k}^2 \quad \text{if}$$

$$(x_a + \tau(x_b - x_a), y_0) \in G_k$$

Therefore

$$\int_{t_0}^t |J'_\alpha(\tau)|^2 d\tau \leq h \int_k |u|_{m, G_k}^2 / h^2 \leq \frac{1}{h} |u|_{m, G_0}^2$$

Condition (e) on the class of admissible grids implies that $1 \leq \frac{C_0}{h}$ and

compactness of G_0 implies that $t \leq C_7$

$$\text{Therefore} \quad |J_\alpha(t)|^2 \leq \frac{C_8^2}{h} |u|_{m, G_0}^2$$

$$\text{and} \quad |||D_\alpha u|||_{0, G_0} = |||J_\alpha|||_{0, Q} \leq \frac{C_8}{h^{\frac{1}{2}}} |u|_{m, G_0} \quad \text{for } |\alpha| = m-1$$

By successive integrations in a similar fashion,

$$|||D_\alpha u|||_{0, G_0} \leq \frac{C_8}{h^{\frac{1}{2}}} |u|_{m, G_0} \quad \text{for } |\alpha| \leq m-1.$$

The lemma follows.

We state without proof the following special form of the Green's

formula. The proof is similar to the one given in reference [1]

Lemma 3. There exist linear differential operators $N_k(x, y, D)$ for $(x, y) \in \bigcup_i \partial G_i$ with coefficients bounded uniformly with respect to all (x, y) and all h , such that N_k is of order $\leq k$ and for all $u \in C^\infty(\bar{G})$ and $\phi \in \mathcal{F}(l, p)$,

$$(1.28) \quad \int_{G_0} \phi Lu = B(\phi, u) + \sum_{S_k \in \partial G} \sum_{j=0}^{m-1} \int_{S_k} (D_{n,j} \phi^{(i_k)} - D_{n,j} \phi^{(i'_k)}) N_{2m-1-j} u d\xi \\ + \sum_{S_k \in \partial G} \sum_{j=0}^{m-1} \int_{S_k} (D_{n,j} \phi^{(i_k)}) N_{2m-1-j} u d\xi$$

where the grid line $S_k = \partial G_{i_k} \cap \partial G_{i'_k}$ or $\partial G \cap \partial G_{i_k}$

Lemma 4. For a given $u \in C^\infty(\bar{G})$

$$(1.28) \quad \left| \int_{G_0} \phi Lu - B(\phi, u) \right| \leq C_{13} h^{p+1} \|\phi\|_{m, G_0} \quad \text{for all } \phi \in \mathcal{F}(l, p) \text{ and all } h$$

Proof: If $\alpha = (\alpha_1, \alpha_2)$, let $\alpha! = \alpha_1! \alpha_2!$ and

$$(x, y)^\alpha = x^{\alpha_1} y^{\alpha_2} \quad \text{Then by Taylors' theorem}$$

$$u = \sum_{|\alpha| \leq p+n} \frac{1}{\alpha!} D^\alpha u(x_0, y_0) (x - x_0)^{\alpha_1} (y - y_0)^{\alpha_2} + R(x, y)$$

where $|||D^\alpha R(x,y)|||_G \leq C_{10} d^{p+m+1-|\alpha|}$ and

$d =$ distance between points (x,y) and (x_0, y_0) .

Now any grid line S_k is given by the equations $x = f_k(\xi_k)$, $y = g_k(\xi_k)$.

Let (x_0, y_0) be a point in S_k . Since f_k and g_k are C^∞ functions and since

the number of shapes of grid lines is finite, it is possible to choose

C_{10} such that

$$(1.29) \quad D_{n,j}^u = \sum_{r=0}^{p+m-j} a_{n,r} \xi_k^r + R_n(\xi_k)$$

where $|||R_n(\xi_k)|||_{s_k} \leq C_{10} \xi_k^{p+m-j+1}$

Substituting (1.29) in Eq. (1.27)

$$\begin{aligned}
\int_{G_0} |\phi Lu - \beta(\phi, u)| &= \sum_{s_k \in \partial G} \left[\sum_{j=0}^{m-1} \sum_{r=0}^{p+j+1-m} b_{nj}^k (\phi_{rj}^{1,k} - \phi_{rj}^{1',k}) + \int_{s_k} (D_{n,j} \phi^{1,k} - D_{n,j} \phi^{1',k}) R_{k,j}'(\epsilon_j) \right. \\
&\quad \left. + \sum_{s_k \in \partial G} \left[\sum_{j=0}^{m-1} \left\{ \sum_{r=0}^{p+j+1-m} b_{nj}^k \phi_{rj}^{1,k} + \int_{s_k} D_{n,j} \phi^{1,k} R_{k,j}'(\epsilon_k) \right\} \right] \right]
\end{aligned}$$

$$\text{when } |||R_{k,j}'(\epsilon_k)|||_{s_k} \leq C_{10} \epsilon_k^{p+j+2-m}$$

Since ϕ satisfies the compatibility conditions (1.8),

$$\phi_{rj}^{1,k} = \phi_{rj}^{1',k} \quad \text{if} \quad s_k = \partial G_{i_k} \cap \partial G_{i_k'}$$

$$\phi_{rj}^{1,k} = 0 \quad \text{if} \quad s_k = \partial G_{i_k} \cap \partial G$$

By Cauchy-Schwarz Inequality,

$$\begin{aligned}
& \left\{ \sum_{s_k \in \partial G} \int_{s_k} (D_{n,j} \phi^{i_k} - D_{n,j} \phi^{i'_k})^2 \right. \\
& \leq \left[\sum_{s_k \in \partial G} \int_{s_k} (D_{n,j} \phi^{i_k} - D_{n,j} \phi^{i'_k})^2 d\xi_k \right] \left[\sum_{s_k \in \partial G} \int_{s_k} R_{kj}^{\prime 2} d\xi_k \right] \\
& \leq C_1 C_{10} \left[h^{2(m-1-j)} \left\{ \sum_{s_k \in \partial G} \int_{s_k} (|\phi|_{m,G_{i_k}}^2 + |\phi|_{m,G_{i'_k}}^2) \right\} \right] \left[\sum_{s_k} \int_{s_k} \xi_k^{2(p+j+2-m)} d\xi_k \right]
\end{aligned}$$

(using Inequalities (1.23), (1.24))

$$\begin{aligned}
& \leq 2C_1 C_{10} \left[h^{2(m-1-j)} h |\phi|_{m,G_0}^2 \right] \left[h^{2(p+j+2-m)} h \frac{(\text{area of } G)}{d^2 h^2} \right] \\
& \leq C_{11} h^{2(p+1)} |\phi|_{m,G_0}^2
\end{aligned}$$

Similarly

$$\left\{ \sum_{s_k \in \partial G} \int_{s_k} D_{nj} \phi^{i_k} R_{kj}' d\xi_k \right\}^2 \leq C_{11} h^{2(p+1)} |\phi|_{m,G_0}^2$$

Therefore
$$\left| \int_{G_0} \phi Lu - B(\phi, u) \right| \leq C_{13} h^{p+1} |\phi|_{m,G_0}$$

Theorem 1: Equation (1.21) has a unique solution $\hat{u}_\varepsilon \mathcal{P}(h,p)$ and

$$(1.31) \quad |||\hat{u}-\bar{u}|||_{m-1,G_0} \leq C_{26} h^{p+1/2}$$

$$(1.32) \quad |\hat{u}-\bar{u}|_{m,G_0} \leq C_{27} h^{p+1}$$

Proof: Suppose for $\varphi, \hat{u}_\varepsilon \mathcal{P}(h,p)$, $B(\varphi, \hat{u}) = 0$ for all φ .

$$\therefore 0 = |B(\hat{u}, \hat{u})| \leq C |\hat{u}|_{m,G_0}^2$$

$$\therefore |||\hat{u}|||_{m-1,G_0} = 0 \text{ by Inequality (1.26)}$$

$$\therefore \hat{u} = 0$$

$$\therefore \text{Eq. (1.21) has a unique } \hat{u}_\varepsilon \mathcal{P}(h,p).$$

Using Taylor's theorem, within each cell, we can write the exact solu-

tion as $\bar{u} = 'u(x_i, y_i) + R(x_i, y_i)$ where $'u(x_i, y_i)$ is a polynomial of degree

$$n_i \text{ and } ||D_\alpha R(x_i, y_i)||_{0,G_i} \leq C_{14} h^{1+n_i-|\alpha|} \text{ for } |\alpha| \leq n_i$$

$$\text{Let } ''U_{kl}^{ij} = \bar{U}_{kl}^{ij} - \bar{U}_{kl}^{ij} \text{ and } ''U_\alpha^{i,w} = \bar{U}_\alpha^{i,w} - U_\alpha^{i,w}$$

$$\text{Then } |''U_{kl}^{ij}| \leq C_{14} h^{n_i+1} \text{ and } |''U_\alpha^{i,w}| \leq C_{14} h^{n_i+1}$$

$$\text{Let } \left\{ u_{kl}^{ij}, u_{\alpha}^{i,w} \right\}_{\text{all } j,w,k,l,\alpha} = d_{\varepsilon} V_1$$

$$\text{Let } a = \bar{T}_1^{-1}(d) = (a_{00}, \dots, a_{0,n_i}).$$

$$\text{Then } |A_{ij}| \leq C_{15} \|d\| \leq C_{16} h^{n_i+1}$$

$$\text{Let } u(x_i, y_i) = a_{00} + a_{10} x_i + \dots + a_{0,n_i} y_i^{n_i}$$

$$\text{where } a_{i,j} = A_{ij}/h^{i+j}$$

$$\text{Then } \|D_{\alpha} u\|_{0,G_1} \leq C_{17} h^{l+n_i-|\alpha|} \quad \text{for } |\alpha| \leq n_i$$

$$\text{Let } {}^0u = 'u + ''u \quad \text{and} \quad \tilde{u} = \bar{u} - {}^0u. \quad \text{Then } {}^0u \in \mathcal{F}(h,p)$$

$$\text{and } \|D_{\alpha} \tilde{u}\|_{0,G_1} \leq C_{18} h^{l+n_i-|\alpha|}, \quad |\alpha| \leq n_i$$

$$\text{Let } \Delta u = \hat{u} - {}^0u \in \mathcal{F}(h,p)$$

$$\begin{aligned} \therefore B(\Delta u, \hat{u}) &= \int_{G_0} f \Delta u = \int_{(\Delta u)} (L\hat{u}) \\ &= B(\Delta u, {}^0u + \tilde{u}) + \left\{ \int_{(\Delta u)} (L\tilde{u}) - B(\Delta u, \bar{u}) \right\} \end{aligned}$$

By Inequality (1.29)

$$|B(\Delta u, \Delta u)| \leq |B(\Delta u, \tilde{u})| + C_{13} h^{p+1} |\Delta u|_{m,G_0}$$

Now the Cauchy-Schwarz Inequality implies that

$$(1.33) \quad |B(\Delta u, \tilde{u})| \leq C_{19} \|\Delta u\|_{m, G_0} \|\tilde{u}\|_{m, G_0} \leq C_{20} h^{1+n_1-m} \|\Delta u\|_{m, G_0}$$

Therefore $|B(\Delta u, \tilde{u})| \leq C_{21} h^{(p+1)} \|\Delta u\|_{m, G_0}$ by Inequalities (1.10) and (1.26).

Inequalities (1.2) and (1.33) imply that

$$\|\Delta u\|_{m, G_0} \leq C_{24} h^{(p+1)}$$

Inequality (1.26) implies that

$$\|\|\Delta u\|\|_{m-1, G_0} \leq C_{25} h^{p+\frac{1}{2}}$$

$$\|\|\hat{u}-\bar{u}\|\|_{m-1, G_0} \leq \|\|\Delta u-\tilde{u}\|\|_{m-1, G_0} \leq \|\|\Delta u\|\|_{m-1, G_0} + \|\|\tilde{u}\|\|_{m-1, G_0} \leq C_{26} h^{p+\frac{1}{2}}$$

and $\|\hat{u}-\bar{u}\|_{m, G_0} \leq C_{27} h^{(p+1)}$

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